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Low-dimensional structures

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Outline

Quantization in nanostructure

Dimensional quantization and low-dimensional Structures

Quantum states of electron in low dimensional structure

Double-quantum-dot structures (artificial molecules)

The first quantity, $N(E)$, is the *number of quantum states* corresponding to the energy interval $(0, E)$.

The second quantity, $g(E)$, called the *density of states*, defines the number of quantum states, N , corresponding to the unit energy interval in the vicinity of E .

For the analysis of electron behavior in a crystal in an external field an approximate method, which is called the *effective-mass approximation*, is widely used.

In order to calculate these quantities we consider first a three-dimensional potential well, which represents bulk crystal.

Such a structure is called *three-dimensional* because of the number of dimensions along which the electron can freely move.

In such a case the electron momentum space is also three-dimensional.

In a crystal with linear dimensions $L_x, L_y,$ and $L_z,$ where they can freely move, the boundary conditions

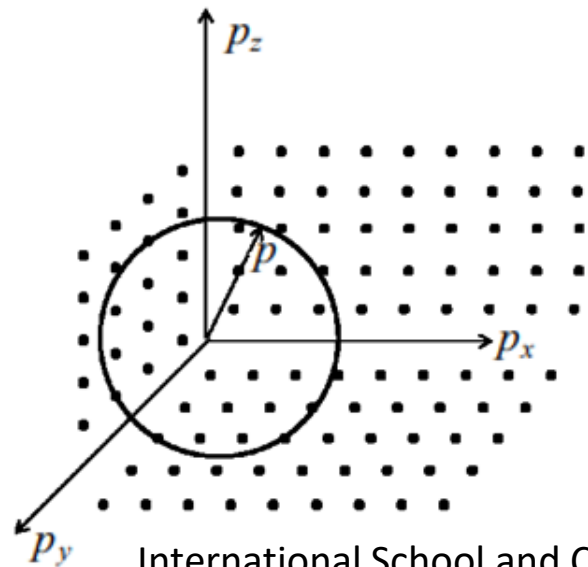
$$\psi(0, y, z) = \psi(x, 0, z) = \psi(x, y, 0) = \psi(L_x, y, z) = \psi(x, L_y, z) = \psi(x, y, L_z) = 0.$$

$$p_x = \frac{\pi \hbar}{L_x} n_x, \quad p_y = \frac{\pi \hbar}{L_y} n_y, \quad p_z = \frac{\pi \hbar}{L_z} n_z,$$

In the case of large $L_x, L_y,$ and L_z the spacing between components of the momentum, is small and we can consider the momentum, $p,$ to be *quasicontinuous*.

The total number of states in the three-dimensional interval of momentum space is $dn = \frac{L_x L_y L_z}{(2\pi \hbar)^3} dp_x dp_y dp_z$

$$dn(p) = \frac{V}{2\pi^2 \hbar^3} p^2 dp.$$



The total number of states $N(p) = 2 \int dn(p) = \frac{V}{\pi^2 \hbar^3} \int_0^p p^2 dp = \frac{V}{3\pi^2 \hbar^3} p^3$

density of states, $g(p) = \frac{dN(p)}{dp} = \frac{p^2 V}{\pi^2 \hbar^3}.$

In many cases it is more convenient to consider energy, E , rather than momentum, \mathbf{p} , as a variable.

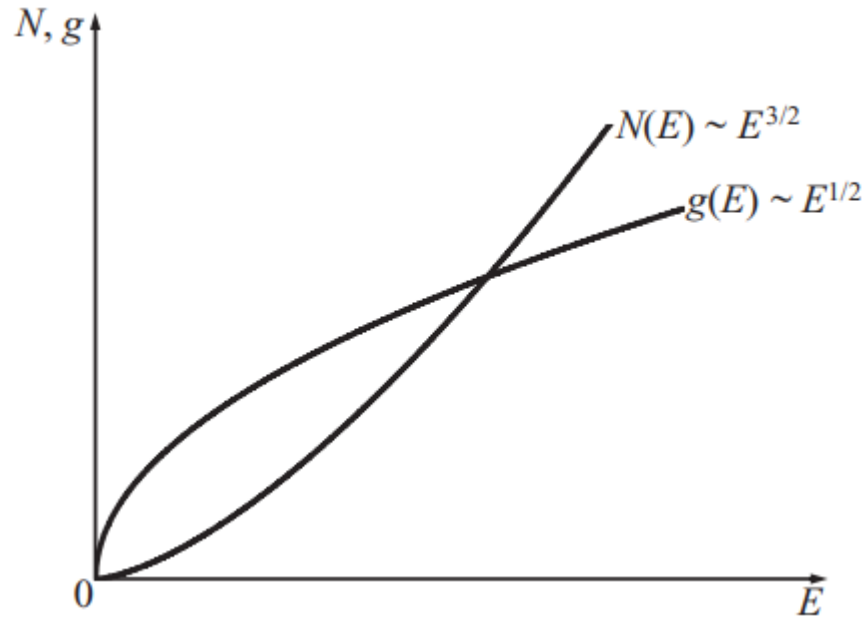
$$E = \frac{p^2}{2m^*}, \quad dE = \frac{p}{m^*} dp, \quad dp = \sqrt{\frac{m^*}{2E}} dE.$$

$$dn(E) = \frac{\sqrt{2}Vm^{*3/2}}{2\pi^2\hbar^3} \sqrt{E} dE.$$

The total number of quantum states in the interval $(0, E)$ is defined by the integral $N(E) = 2 \int dn(E) = 2 \frac{\sqrt{2}Vm^{*3/2}}{3\pi^2\hbar^3} E^{3/2}$.

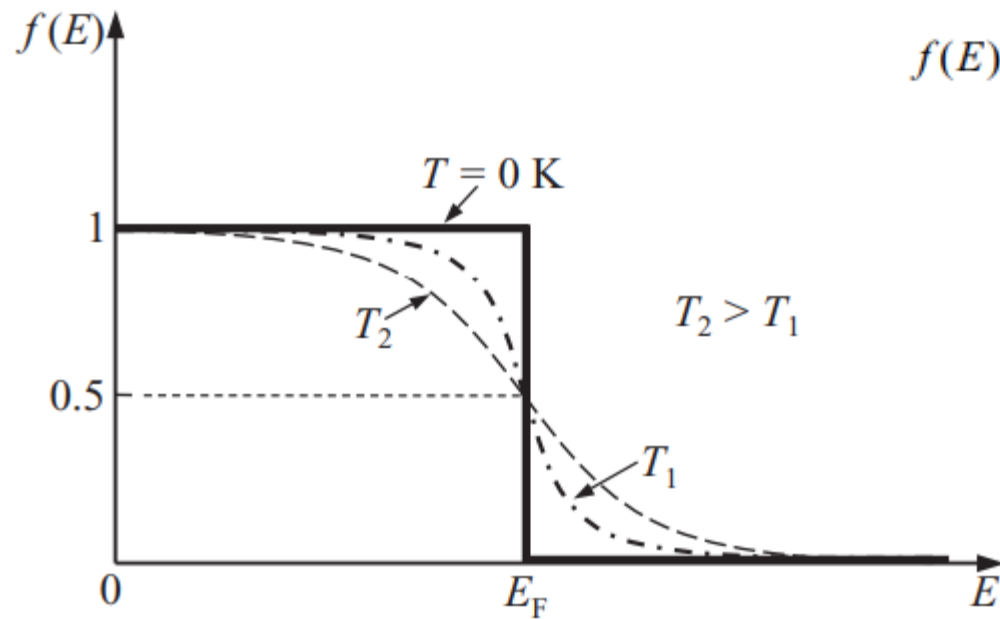
For the density of states, $g(E)$, i.e., the number of states that corresponds to the unit energy interval,

$$g(E) = \frac{dN(E)}{dE} = \frac{\sqrt{2}Vm^{*3/2}}{\pi^2\hbar^3} \sqrt{E}.$$



dependences of the number, $N(E)$, and the density of states, $g(E)$, on energy, E , in the case of an electron's three-dimensional motion in a crystal.

The probability of finding an electron in the state with energy E is given by the *Fermi–Dirac distribution function*, $f(E)$:



$$f(E) = \frac{1}{e^{(E-E_F)/(k_B T)} + 1}$$

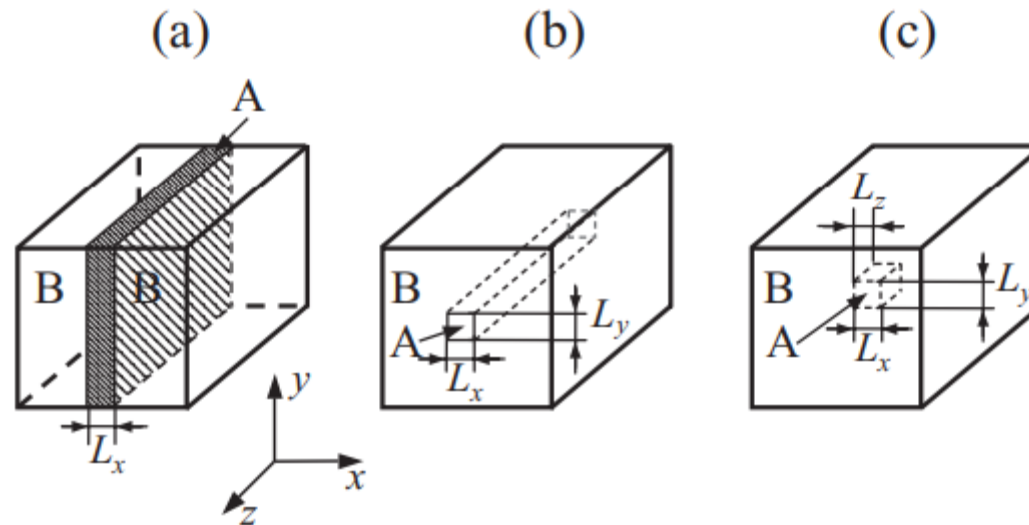
The Fermi–Dirac distribution function, $f(E)$, for various temperatures. The distribution functions at temperatures higher than 0 K are shown by dashed and dash-dotted lines. Here, $T_2 > T_1$.

$$N = N(E_F) = 2 \frac{\sqrt{2} V m^{*3/2}}{3\pi^2 \hbar^3} E_F^{3/2}$$

$$E_F = \frac{\hbar^2}{2m^*} \left(\frac{3\pi^2 N}{V} \right)^{2/3}$$

Let us estimate E_F for the conduction electrons in a metal, for example in copper. For copper the electron effective mass is close to the free electron mass: $m^* \approx m_e$. The electron density for copper is equal to $\rho = N/V \approx 10^{29} \text{ m}^{-3}$. Thus, the Fermi energy is about several electron-volts and the distance between energy levels is infinitesimally small compared with E_F . Therefore, the energy spectrum of the electron gas in metals can be considered to be *quasicontinuous*.

Different types of nanostructures: (a) a quantum well, (b) a quantum wire, and (c) a quantum dot. A and B denote the materials which constitute the nanostructures, with A being the material of the nanostructure itself and B being the material of the surrounding matrix. The nanostructure is referred to as *free-standing* if the surrounding matrix, B, is vacuum.



a space quantization:

$$\lambda^* = \frac{h}{p} = \frac{2\pi\hbar}{m^*v} = \frac{2\pi\hbar}{\sqrt{3m^*k_B T}}$$

$$L_x \leq \lambda^*$$

quantum well (QW).

$$L_x \leq \lambda^* \quad \text{and} \quad L_y \leq \lambda^*$$

quantum wire (QWR)

$$L_x \leq \lambda^*, \quad L_y \leq \lambda^*, \quad \text{and} \quad L_z \leq \lambda^*$$

quantum dot (QD)

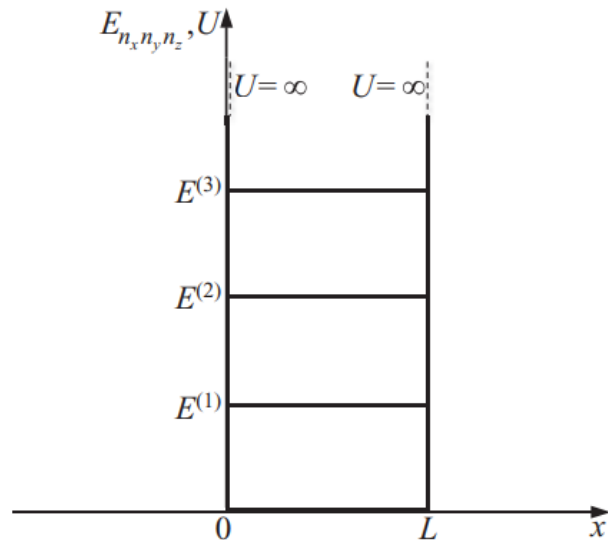
In order to observe the quantization of the energy spectrum experimentally, the distance between the adjacent energy levels has to be sufficiently large. First of all it must be larger than the thermal energy of electrons in a quantum well, i.e.,

$$E_{n+1} - E_n > k_B T.$$

This condition must be satisfied in order to exclude the possible thermal transitions of the electron between quantized energy levels which will hinder the observation of the *quantum-dimensional effects* (QDEs).

QDEs the mean free path of the electron, l_e , in the medium must be substantially larger than the size of the region of quantized motion. This is because the quantization takes place only if the electron wavefunction has the form of a standing wave in the region of electron motion. Such a regularity of the wavefunction is possible only in the case of weak electron scattering on vibrations of atoms of the films as well as on defects, which are always present in real structures and destroy the coherent character of the electron motion.

Energy levels in a quantum dot.



$$E_{n_x n_y n_z} = \frac{\pi^2 \hbar^2}{2m^* L^2} (n_x^2 + n_y^2 + n_z^2)$$

The ground state $n_x = n_y = n_z = 1$:

$$E^{(1)} = E_{\min} = E_{111} = \frac{3\pi^2 \hbar^2}{2m^* L^2}.$$

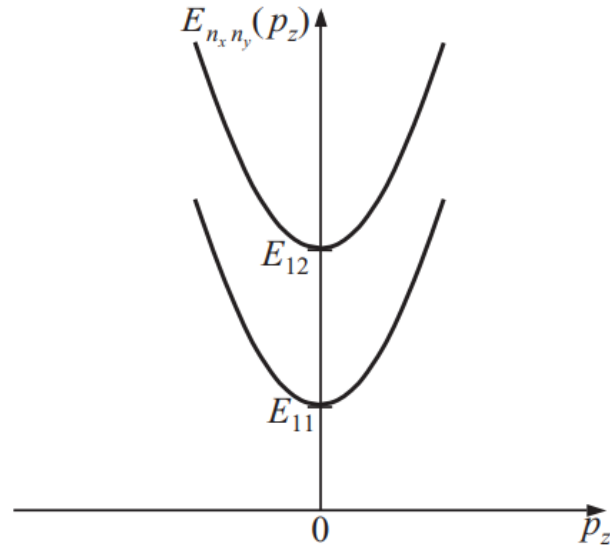
sets of quantum numbers, (112, 121, 211),

$$\psi_{n_x n_y n_z}(x, y, z) = \sqrt{\frac{8}{L^3}} \sin\left(\frac{\pi n_x x}{L}\right) \sin\left(\frac{\pi n_y y}{L}\right) \sin\left(\frac{\pi n_z z}{L}\right)$$

$$E^{(2)} = E_{112} = E_{121} = E_{211} = \frac{3\pi^2 \hbar^2}{m^* L^2}.$$

Degeneracy

Energy spectrum in a quantum wire

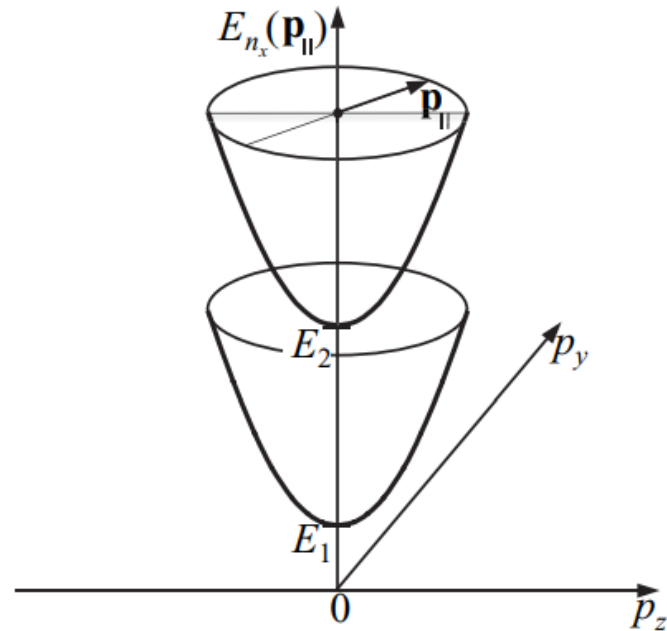


Quantum wire:
the quantized energy
levels, E_{11} and E_{12} , as
well as the dependences
of the electron energy
 $E_{n_x n_y}$ on the momentum p_z

$$\psi_{n_x n_y}(x, y, z) = \sqrt{\frac{4}{L_x L_y}} \sin\left(\frac{\pi n_x x}{L_x}\right) \sin\left(\frac{\pi n_y y}{L_y}\right) e^{ik_z z},$$

$$E_{n_x n_y}(k_z) = \frac{\hbar^2}{2m^*} \left(\frac{\pi^2 n_x^2}{L_x^2} + \frac{\pi^2 n_y^2}{L_y^2} \right) + \frac{\hbar^2 k_z^2}{2m^*},$$

Energy spectrum in a quantum well



Quantum well:
quantized energy levels,
 E_1 and E_2 , and the
dependences of the
electron energy, E_{n_x} , on
the momentum

$$|\mathbf{p}_{\parallel}| = \hbar \sqrt{k_y^2 + k_z^2}$$

$$\psi_{n_x}(x, y, z) = \sqrt{\frac{2}{L_x}} \sin\left(\frac{\pi n_x x}{L_x}\right) e^{i(k_y y + k_z z)},$$

$$E_{n_x}(k_y, k_z) = \frac{\hbar^2}{2m^*} \left(\frac{\pi^2 n_x^2}{L_x^2}\right) + \frac{\hbar^2(k_y^2 + k_z^2)}{2m^*}.$$

Number of States and Density of States for Quantum Well

$$U(x) = \begin{cases} 0, & 0 \leq x \leq L_x, \\ \infty, & x < 0, x > L_x. \end{cases}$$

The region of quantization is $0 \leq x \leq L_x$.

electron motion can be considered *quasi-two-dimensional*.

$$E_{n_x}(k_y, k_z) = \frac{\hbar^2}{2m^*} \left(\frac{\pi^2 n_x^2}{L_x^2} \right) + \frac{\hbar^2 (k_y^2 + k_z^2)}{2m^*},$$

$$E_{n_x}(p_y, p_z) = \frac{\hbar^2}{2m^*} \left(\frac{\pi^2 n_x^2}{L_x^2} \right) + \frac{p_y^2 + p_z^2}{2m^*}.$$

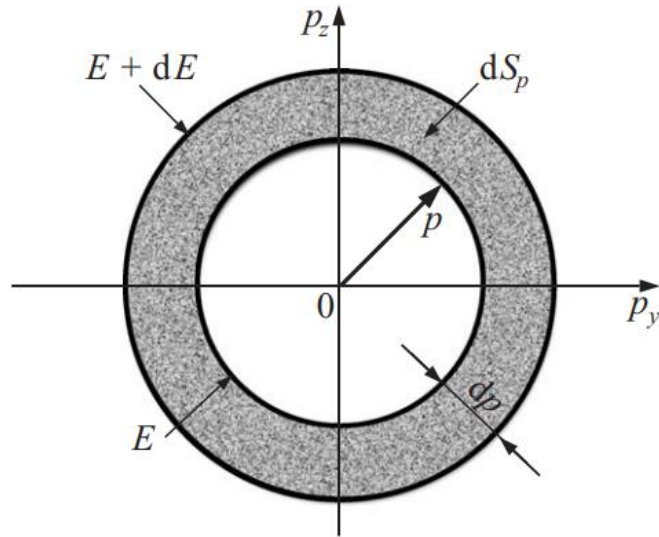
The lowest energy in each of the subbands $p_y = p_z = 0$ $E_{n_x} = E_{n_x}(0, 0) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{n_x}{L_x} \right)^2$

The lowest energy that an electron can have $n_x = 1$ $E_1 = E_1(0, 0) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{1}{L_x} \right)^2$

Electron states with $E < E_1$ are forbidden. Therefore, for $E < E_1$ $N(E) = 0$, and $g(E) = 0$.

The lowest energy value in the second subband $E_2 = E_2(0, 0) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{2}{L_x} \right)^2$

In the energy region $E_1 < E < E_2$, there are states that belong only to the first subband, $E_1(p_y, p_z)$.



A surface of equal energy, E , in a two-dimensional \mathbf{p} -space for a given n_x .

The number of quantum states corresponding to this area, dS_p , and $S_r = L_y L_z$, $dn(p) = \frac{S_r \times 2\pi p dp}{(2\pi \hbar)^2} = \frac{S_r p dp}{2\pi \hbar^2} = \frac{S_r m^* dE}{2\pi \hbar^2}$

The total number of states in the energy interval $(0, E)$

$$N_0(E) = 2 \int dn(E) = 2 \int_0^E \frac{S_r m^*}{2\pi \hbar^2} dE = \frac{m^* L_y L_z}{\pi \hbar^2} E$$

The density of states,

$$g_0(E) = \frac{dN_0(E)}{dE} = \frac{m^* L_y L_z}{\pi \hbar^2}.$$

In the energy region $E_1 < E < E_2$,

$$N(E) = N_0(E - E_1) = \frac{m^* L_y L_z}{\pi \hbar^2} (E - E_1),$$

$$g(E) = \frac{dN(E)}{dE} = \frac{m^* L_y L_z}{\pi \hbar^2}.$$

In the energy interval $E_2 < E < E_3$ the number and density of states are equal to the sum of states of the first two subbands,

$$N(E) = N_0(E - E_1) + N_0(E - E_2) = \frac{m^* L_y L_z}{\pi \hbar^2} (2E - E_1 - E_2),$$

$$g(E) = \frac{2m^* L_y L_z}{\pi \hbar^2}.$$

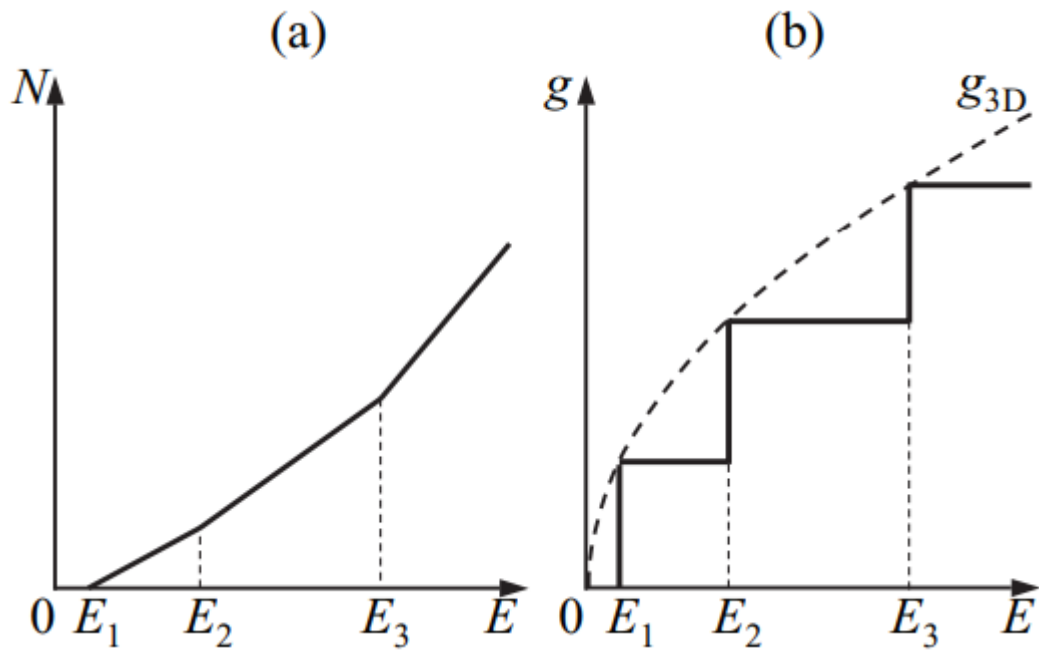
For an energy in the interval $E_n < E < E_{n+1}$

$$N(E) = \frac{m^* L_y L_z}{\pi \hbar^2} (nE - E_1 - E_2 - \dots - E_n),$$

$$g(E) = \frac{nm^* L_y L_z}{\pi \hbar^2}.$$

Thus, the number of states, $N(E)$, is a continuous function consisting of linear segments with slope increasing from region to region.

$$\alpha_n = \arctan \left(\frac{nm^* L_y L_z}{\pi \hbar^2} \right) \quad \text{The density of states, } g(E), \text{ is a step-function, which undergoes jumps, } \Delta g_n = \frac{m^* L_y L_z}{\pi \hbar^2},$$



The dependences of (a) the number of states, $N(E)$, and (b) the density of states, $g(E)$, in a rectangular quantum well with barriers of infinite height. Here g_{3D} is the electron density of states in bulk (three-dimensional) material.

Number of States and Density of States for Quantum Wires

$$U(x, y) = \begin{cases} 0, & 0 \leq x \leq L_x, 0 \leq y \leq L_y, \\ \infty, & x < 0, y < 0, \\ \infty, & x > L_x, y > L_y. \end{cases} \quad E_{n_x n_y}(k_z) = \frac{\hbar^2 \pi^2}{2m^*} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) + \frac{\hbar^2 k_z^2}{2m^*} = \frac{\hbar^2 \pi^2}{2m^*} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) + \frac{p_z^2}{2m^*}.$$

electron motion can be considered *quasi-one-dimensional*.

The minimal energy in each of the subbands corresponds to their bottom with $p_z = 0$: $E_{n_x n_y} = E_{n_x n_y}(0) = \frac{\hbar^2 \pi^2}{2m^*} \left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right)$

The lowest energy which an electron can have in a quantum wire $E_{11} = E^{(1)} = E_{11}(0) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{1}{L_x^2} + \frac{1}{L_y^2} \right)$

Electron states are forbidden for energies lower than $E^{(1)}$ Therefore, for all energies $E < E^{(1)}$ $N(E) = 0$ and $g(E) = 0$.

In the energy interval $E^{(1)} < E < E^{(2)}$ $E^{(2)} = E_{12}(0) = E_{21}(0) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{1}{L_x^2} + \frac{2^2}{L_y^2} \right) = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{2^2}{L_x^2} + \frac{1}{L_y^2} \right)$

there exist states that belong only to the first subband $E_{11}(p_z)$.

z -component of the electron momentum in the corresponding subband $p_z = \sqrt{2m^*[E - E_{n_x n_y}(0)]}$.

$$dp_z = \sqrt{\frac{m^*}{2}} \frac{dE}{\sqrt{E - E_{n_x n_y}(0)}}, \quad p_z = \frac{\pi \hbar n_z}{L_z}, \quad g(E) = \frac{dN(E)}{dE} = \frac{L_z}{\pi \hbar} \sqrt{\frac{2m^*}{E - E_{n_x n_y}(0)}}.$$

In the energy interval $E^{(1)} < E < E^{(2)}$ the number and density of states

$$N(E) = \frac{2L_z}{\pi \hbar} \sqrt{2m^*(E - E_{11})},$$

$$g(E) = \frac{L_z}{\pi \hbar} \sqrt{\frac{2m^*}{E - E_{11}}}.$$

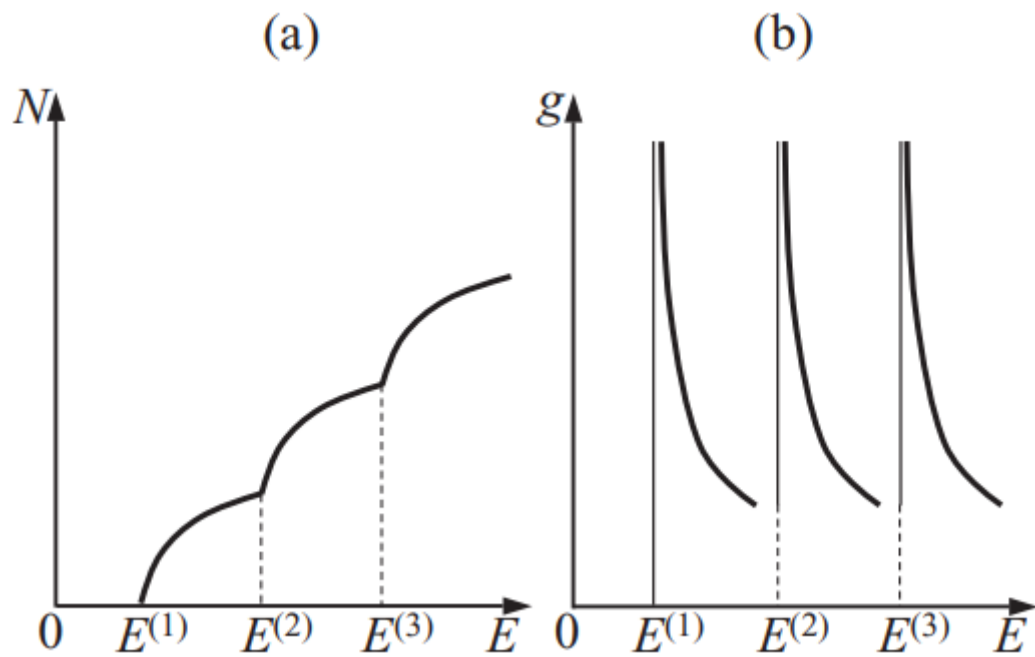
In the general case, for an energy E

Heaviside function Θ (or the *step-function*):

$$N(E) = \frac{2L_z}{\pi \hbar} \sum_{n_x, n_y} \sqrt{2m^*(E - E_{n_x n_y})} \times \Theta(E - E_{n_x n_y}),$$

$$g(E) = \frac{L_z}{\pi \hbar} \sum_{n_x, n_y} \sqrt{\frac{2m^*}{E - E_{n_x n_y}}} \times \Theta(E - E_{n_x n_y}).$$

$$\Theta(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$



The dependences of (a) the number of states, $N(E)$, and (b) the density of states, $g(E)$, in the rectangular quantum wire with barriers of infinite height.

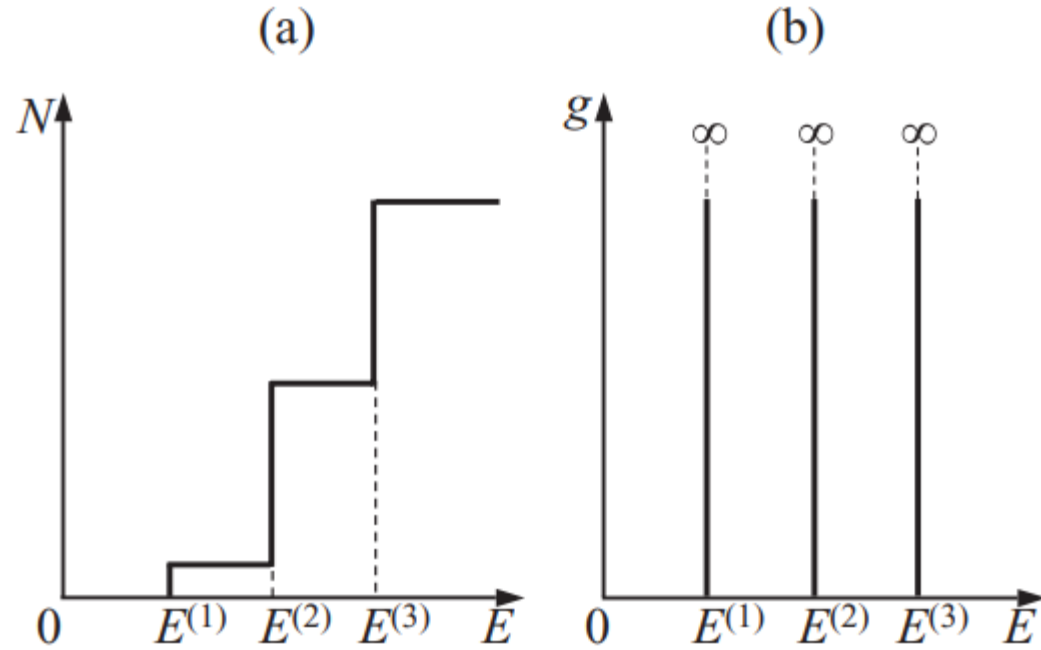
Number of States and Density of States for Quantum Dots

If the quantum dot has the form of a cubic box ($L_x = L_y = L_z$) with potential barriers of infinite height, the energy levels with different n_α are degenerate ($\alpha = x, y, \text{ and } z$). In this case the jump in the number of states, $N(E)$, at $E = E_{n_x n_y n_z}$ is equal to twice the degeneracy, g_{n_x, n_y, n_z} , of the corresponding level.

$$N(E) = 2 \sum_{n_x, n_y, n_z} g_{n_x, n_y, n_z} \times \Theta(E - E_{n_x n_y n_z}),$$

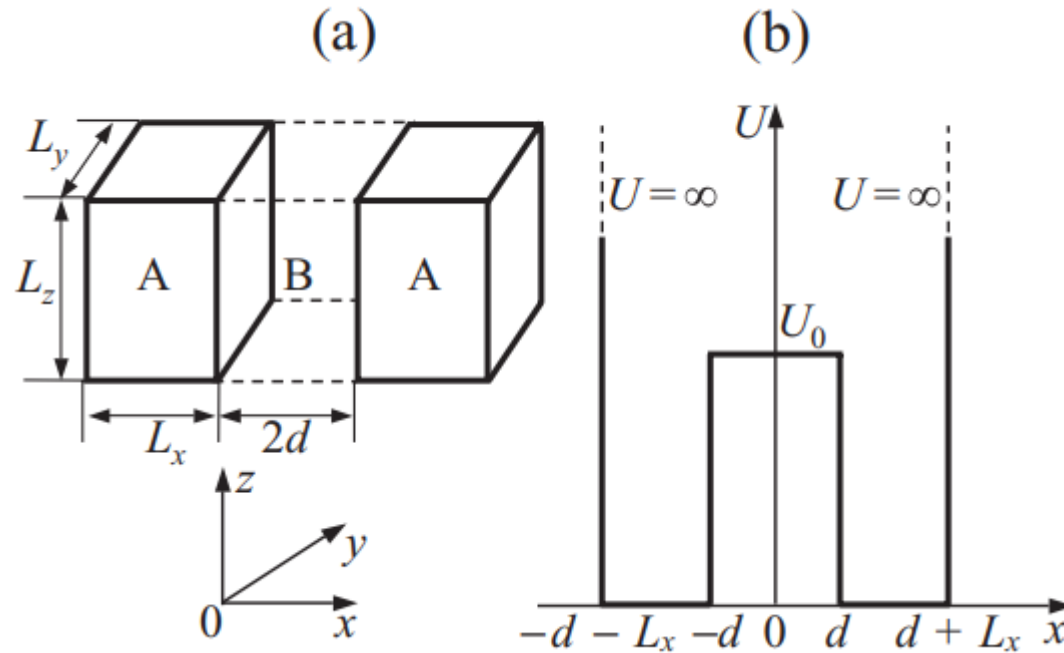
$$g(E) = 2 \sum_{n_x, n_y, n_z} g_{n_x, n_y, n_z} \times \delta(E - E_{n_x n_y n_z}),$$

$\delta(E - E_{n_x n_y n_z})$ is Dirac's δ -function, $\Theta(E - E_{n_x n_y n_z})$ the Heaviside function, and g_{n_x, n_y, n_z} the degeneracy



The dependences of (a) the number of states, $N(E)$, and (b) the density of states, $g(E)$, in a rectangular box with potential barriers of infinite height.

Double-quantum-dot structures (artificial molecules)



A schematic picture of a double-quantum-dot structure (a) and its potential profile $U(x)$ (b). A and B denote different semiconductor materials, which constitute quantum dots and barriers between them.

The general dependence of the potential $U(x, y, z)$

$$U(x, y, z) = \begin{cases} \infty, & y < 0, z < 0, \\ U(x), & 0 \leq y \leq L_y, 0 \leq z \leq L_z, \\ \infty, & y > L_y, z > L_z, \end{cases} \quad U(x) = \begin{cases} U_0, & |x| \leq d, \\ 0, & d \leq |x| \leq d + L_x, \\ \infty, & |x| > d + L_x. \end{cases}$$

In the wavefunction of such a potential we can separate variables and write the total wavefunction as

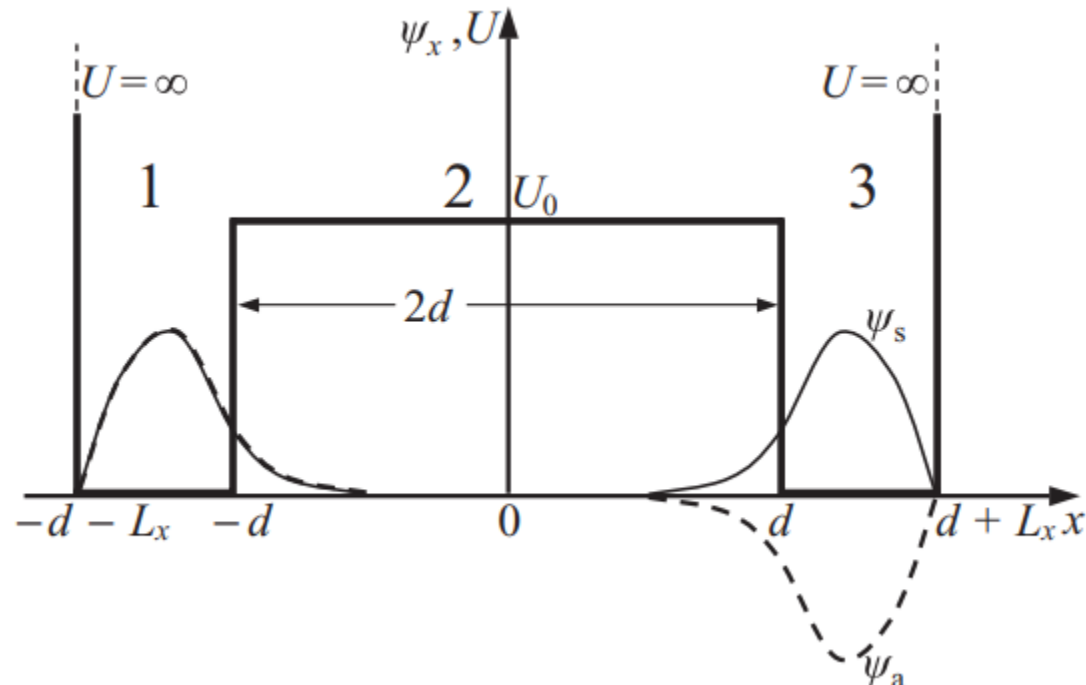
$$\psi(x, y, z) = \psi_x(x)\psi_y(y)\psi_z(z),$$

$$\psi_y(y) = \sqrt{\frac{2}{L_y}} \sin\left(\frac{n_y \pi y}{L_y}\right), \quad \psi_z(z) = \sqrt{\frac{2}{L_z}} \sin\left(\frac{n_z \pi z}{L_z}\right).$$

The x -component of the wavefunction, $\psi_x(x)$, must satisfy the one-dimensional Schrödinger equation with the potential $U(x)$. The total electron energy in the double-quantum-dot structure must be a sum of the energies of quantum confinement along the y - and z -directions and the energy of electron motion along the x -axis, E_x , in the potential $U(x)$:

$$E = E_x + \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2} \right)$$

Qualitative Analyzes

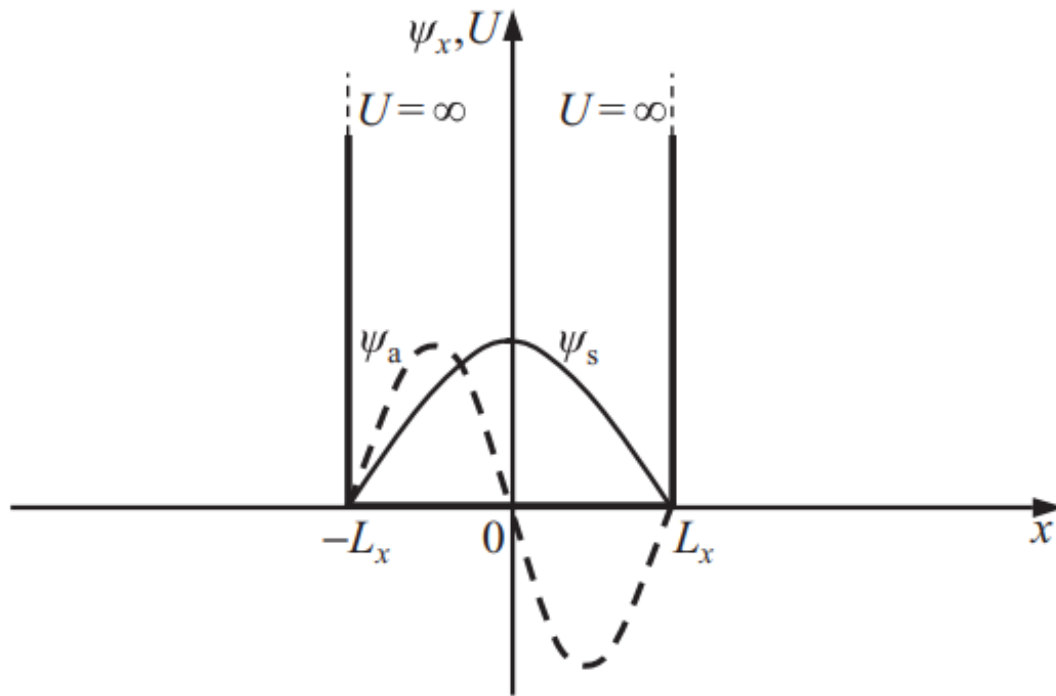


The wavefunctions of a double-quantum-dot structure, ψ_s and ψ_a , when the quantum dots are far from each other.

The state with the symmetric wavefunction, ψ_s , corresponds to a smaller energy E_s than the energy E_a , which corresponds to the state with the antisymmetric wavefunction, ψ_a .

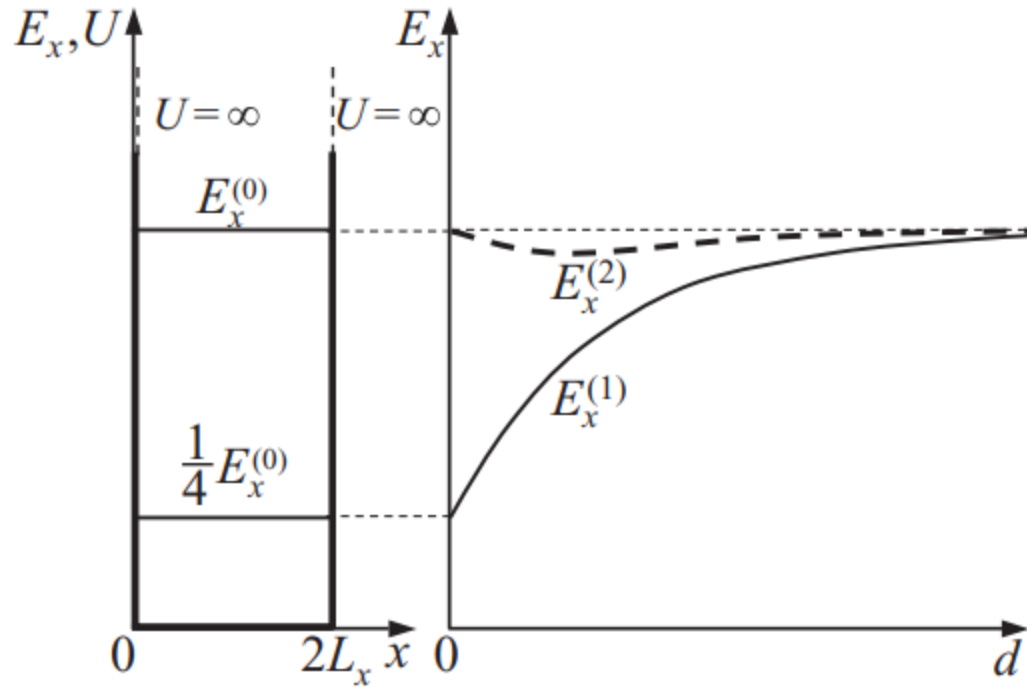
This is due to the fact that the average value of the kinetic energy for the symmetric state is smaller than for the antisymmetric wavefunction.

$$\overline{\left| \frac{d\psi_s}{dx} \right|^2} < \overline{\left| \frac{d\psi_a}{dx} \right|^2}$$



The wavefunctions ψ_s and ψ_a , when the width of the barrier between the two quantum dots is equal to zero.

$$E_x = \frac{\pi^2 \hbar^2}{2m^*(2L_x)^2} n_x^2 = E_x^{(0)} \frac{n_x^2}{4}$$



The dependences of the energies of the symmetric, $E_x^{(1)}$, and antisymmetric, $E_x^{(2)}$, states of the double-quantum-dot structure on the width of the barrier, d , separating individual quantum dots.

Double-quantum-dot structures: exact solution

we assume that the *electron effective mass, m^* , in the potential wells and that in the barriers are the same.*

For the regions 1 and 3, where $d \leq |x| \leq d + L_x$

$$-\frac{\hbar^2}{2m^*} \frac{d^2 \psi_j(x)}{dx^2} = E_x \psi_j(x), \quad k = \sqrt{\frac{2m^* E_x}{\hbar^2}}, \quad j = 1, 3$$

for the region $|x| \leq d$ with $j = 2$

$$-\frac{\hbar^2}{2m^*} \frac{d^2 \psi_2(x)}{dx^2} + U_0 \psi_2(x) = E_x \psi_2(x),$$
$$\psi_2(x) = A_2 e^{\kappa x} + B_2 e^{-\kappa x} \quad \kappa = \sqrt{\frac{2m^*(U_0 - E_x)}{\hbar^2}}.$$

The boundary conditions for this structure are

$$\psi_1(-d - L_x) = \psi_3(d + L_x) = 0, \quad \left(\frac{d\psi_1}{dx} - \frac{d\psi_2}{dx} \right) \Big|_{x=-d} = 0,$$
$$\psi_1(-d) = \psi_2(-d),$$
$$\psi_3(d) = \psi_2(d), \quad \left(\frac{d\psi_3}{dx} - \frac{d\psi_2}{dx} \right) \Big|_{x=d} = 0.$$

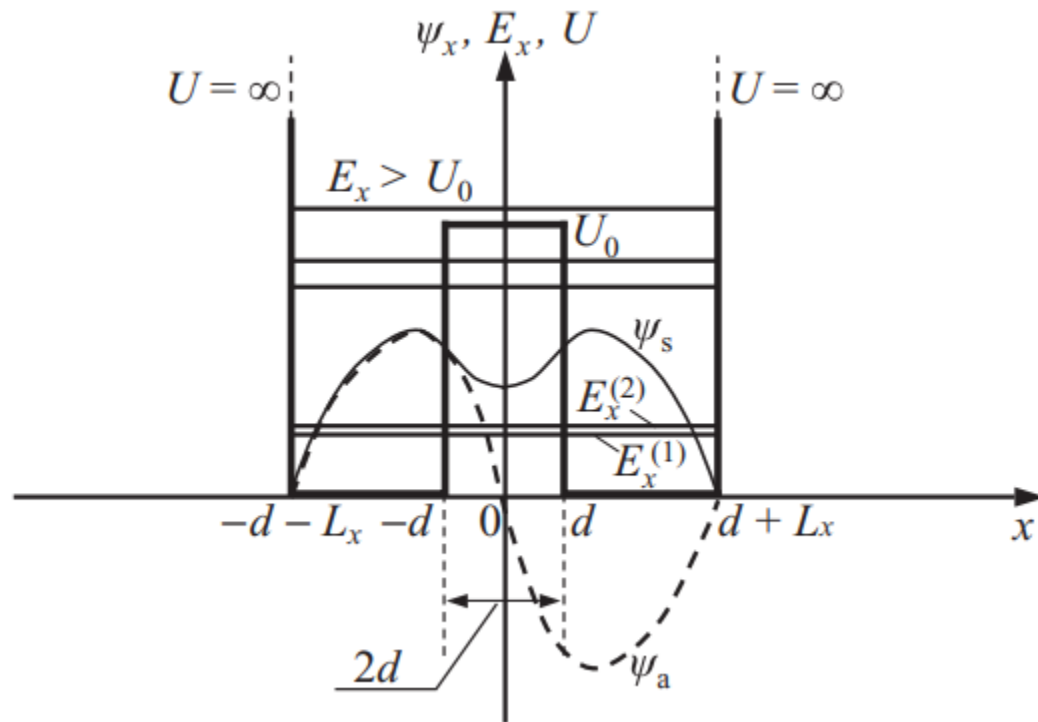
dispersion relations:

$$\begin{aligned} \tan(kL_x) + \frac{k}{\kappa} \coth(\kappa d) &= 0, & \text{symmetric states} \\ \tan(kL_x) + \frac{k}{\kappa} \tanh(\kappa d) &= 0, & \text{antisymmetric states.} \end{aligned} \quad \text{where} \quad \tanh \alpha = \frac{1}{\coth \alpha} = \frac{e^\alpha - e^{-\alpha}}{e^\alpha + e^{-\alpha}}.$$

For sufficiently large distances between the quantum dots, i.e., at $\kappa d \gg 1$

$$\tan(kL_x) \approx -\frac{k}{\kappa} = -\sqrt{\frac{E_x}{U_0 - E_x}}.$$

For the first two deepest levels at $E_x \ll U_0$ $kL_x \approx \pi$ $E_x^{(1,2)} \approx \frac{\pi^2 \hbar^2}{2m^* L_x^2} = E_x^{(0)}$



The wavefunctions of a double-quantum-dot structure, ψ_s and ψ_a , when the quantum dots are close to each other.

In the case of a narrow barrier i.e., at $\kappa d \ll 1$

$$\tan(kL_x) = -\frac{k}{\kappa^2 d}, \quad \text{symmetric states}$$

$$\tan(kL_x) = -\kappa d. \quad \text{antisymmetric states.}$$

$kL_x = \arctan(-\kappa d)$. In a Taylor series taking into account only its first two terms.

$$\arctan t = \pi + t. \quad kL_x = \pi - \kappa d. \quad E_x^{(2)} \approx \frac{\pi^2 \hbar^2}{2m^* L_x^2} \frac{1}{(1 + d/L_x)^2}.$$

We see that at $d = 0$ the energy of the antisymmetric state becomes equal to the energy of the first excited level of an individual quantum dot with width $2L_x$

$$E_x^{(2)} \approx \frac{\pi^2 \hbar^2}{2m^* L_x^2} = E_x^{(0)}.$$

$$\text{In the limiting case when } d = 0, \quad kL = \frac{\pi}{2} \quad E_x^{(1)} = \frac{E_x^{(0)}}{4}.$$

Let us consider now the case when ($E_x > U_0$).

The solutions of the Schrödinger equation for regions 1 and 3 have the same form and the solution in region 2 can be written as

$$\psi_2(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}, \quad \text{where} \quad k_2 = \sqrt{\frac{2m^*(E_x - U_0)}{\hbar^2}}.$$

$$\tan(kL_x) - \frac{k}{k_2} \cot(k_2 d) = 0, \quad \text{symmetric states}$$

these are *transcendental equations*.

$$\tan(kL_x) + \frac{k}{k_2} \tan(k_2 d) = 0, \quad \text{antisymmetric states.}$$

Let us assume that in the region of the quantum dots, 1 and 3, as well as in the region of the barrier, 2, the electron waves are de Broglie standing waves.

$$k_2 d = m\pi$$

$$k_2 d = m\pi$$

symmetric states

antisymmetric states.

$$kL_x = \left(n + \frac{1}{2}\right) \pi$$

$$kL_x = n\pi$$

$$E_x^s = \frac{\pi^2 \hbar^2}{2m^* L_x^2} \left(n + \frac{1}{2} \right)^2 = U_0 + \frac{\pi^2 \hbar^2}{2m^* d^2} m^2. \quad U_0 = \frac{\pi^2 \hbar^2}{2m^*} \left[\frac{(n + 1/2)^2}{L_x^2} - \frac{1}{d^2} \right]$$

For the energy of antisymmetric states,

The lowest energy value corresponds to $m = 1$.

$$E_x^a = \frac{\pi^2 \hbar^2}{2m^* L_x^2} n^2 = U_0 + \frac{\pi^2 \hbar^2}{2m^* d^2} m^2. \quad U_0 = \frac{\pi^2 \hbar^2}{2m^*} \left(\frac{n^2}{L_x^2} - \frac{1}{d^2} \right)$$

The realization of the above-barrier regime with symmetric or antisymmetric states, defined by quantum numbers n and m , is possible only if one of the following conditions is satisfied:

$$d > \frac{m}{n + 1/2} L_x, \quad \text{or} \quad d > \frac{m}{n} L_x.$$

Thank you very much for your attention